The Armchair Space Traveler

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Abstract
This is an article presents a vicarious adventure into space through the examination of various problems in physics. The first takes the armchair traveler to the moon. Upon arrival there, the second problem concerns a study of elements in a sample taken from the moon's surface. The next move is a journey to Jupiter as the third problem studies the "slingshot" effect whereby the gravity of Jupiter is used to accelerate a spaceship past that planet. Finally, our armchair journey takes a cosmic leap with the final problem dealing with the use of a fusion reaction to gain more acceleration for a trip to the stars.

1. Introduction

The Age of Space Travel was inaugurated in 1957 when Russian engineers placed their first Sputnik in orbit around the earth. Shortly thereafter, they sent a dog around the world aboard their second Sputnik. Thus the first space passenger to orbit the Earth was canine rather than human. Then, in 1961, the Russian cosmonaut Yuri Gagarin was launched into a single orbit, thus qualifying as the first human space traveler (of whom we are aware). The first space venture other than "around the World and back again" occurred in 1965 when the unmanned Russian craft Venera 3 crashed onto the surface of Venus. This flight was followed in 1966 by the soft landing of the American Surveyor 1 on the Moon.

True space travel in which a human being visited a body in space other than the Earth began in 1969 with the flight of the American Apollo 11 which carried Neil Armstrong, Buzz Aldrin, and Michael Collins to the Moon. While Collins remained behind circling the Moon in the command module, the lunar landing craft carried Armstrong and Aldrin to the lunar surface where they safely disembarked to take their triumphal walks. The world watched the space adventure and the safe return of the three astronauts on its television screens.

Just as the expeditions of the Nineteenth Century and early Twentieth Century explorers who traveled to the unknown regions of planet Earth stimulated interest in both real and vicarious terrestrial travel, so the voyages of the first cosmonauts and astronauts have inspired actual and armchair space travel. Armchair space travel is the safer and more comfortable of the two varieties, but with paper, pencil, and calculator, we can investigate and solve problems that might confront real space travelers while remaining in the comfort of our own studies. Thus armchair space travel can possess a reality beyond the reading of descriptions of journeys to the source of the Nile, or treks across Antarctica to the South Pole.

2. The Problems

The first problem has three parts.
Problem 1. (Flying to the Moon)

A. What is the escape velocity for a rocket launched from the surface of the Earth?
B. What is the minimum energy per kilogram required to deliver a spacecraft to the Moon from the surface of the Earth?
C. How much kinetic energy per kilogram must be dissipated (by retro-rockets) in order that the spacecraft make a soft landing on the lunar surface?

Before proceeding with our solutions, we list the numerical values in S.I. units of the physical constants that will appear in our calculations:

g = gravitational acceleration at the Earth's surface = 9.81
G = universal gravitational constant = 6.672 x 10^{-11}
$R(0)$ = distance between the centers of the Earth and Moon = 3.844 x 10^{8}
$R(E)$ = radius of the Earth = 6.37 x 10^{6}
$R(M)$ = radius of the Moon = 1.738 x 10^{6}
M = mass of the Earth = 5.974 x 10^{24}
m = mass of the Moon = 0.735 x 10^{24}

Solution.
A. Let us assume that all of the rocket's kinetic energy T is delivered to it at a single shot at the instant of lift-off and that T is sufficient to carry the rocket beyond the Earth's gravitational field (i.e. to "infinity"). Furthermore, let us assume that air resistance does not act on the rocket as it travels through the Earth's atmosphere. Then

$$T = \left(\frac{1}{2}\right)\mu v^2 \geq \frac{GM\mu}{R(E)} - 0 = \frac{GM\mu}{R(E)}$$

where μ and v represent the mass of the rocket after its fuel has been spent, and the initial velocity of the rocket, respectively.

If v is taken to be the escape velocity, then

$$\left(\frac{1}{2}\right)\mu v^2 = \frac{GM\mu}{R(E)} \qquad (1)$$

Equation (1) states that all of the kinetic energy delivered to the rocket at lift-off is converted to gravitational potential energy. Since $g = \frac{GM}{R(E)^2}$, equation (1) may be rewritten as $v^2 = 2gR(E)$. Solving this equation for v and using the values given above for the relevant constants, we find the escape velocity to be $v \approx 11.19\ ^{km}/_{s}$.

B. Since the spacecraft does not have to leave the earth's gravitational field, but only travel to the Moon, and since the Moon's gravitational pull on the spacecraft will be helpful in the flight, the kinetic energy required to send the spacecraft from the surface of the Earth to the surface of the Moon is less than that associated with the escape velocity.
To simplify our thinking, les us imagine that the spacecraft travels along the line of centers of the Earth and the Moon. We then note that there is a point, say P, on this line at which the gravitational attractions by the Earth and the Moon precisely cancel one another out. Now let the distance of P from the center of the Earth be denoted by r. Since the gravitational forces by the Earth and the Moon on the spacecraft are of equal size but in opposite directions at P, we may write that $\frac{GM}{r^2} = \frac{Gm}{(R(0)-r)^2}$ which implies that $\frac{R(0)}{r} = 1 + \sqrt{m/M} \approx 1.111$. Thus $r \approx 3.46 \times 10^8$ m.

The work performed on 1 kilogram of mass in carrying it from the surface of the Earth to point P is given by

$$T' = \int_{R(E)}^{r} \left(\frac{GM}{x^2} - \frac{GM}{(R(0)-x)^2}\right) dx = \left(\frac{-GM}{r} + \frac{GM}{R(E)}\right) - \left(\frac{GM}{R(0)-r} - \frac{GM}{R(0)-R(E)}\right) \qquad (2)$$

Evaluating the right hand side of equation (2) by substituting the value of r as found above and the values of the physical constants listed at the beginning of our solution yields the kinetic energy converted to gravitational potential energy as 1 kilogram is lifted from the surface of the Earth to point P. Any kinetic energy in excess of T' will carry the kilogram to P and, thereafter, the kilogram will plummet to the lunar surface under the gravitational attraction of the Moon. The value of T' may be taken to be the minimum value of initial kinetic energy per kilogram which will send the spacecraft to the Moon. We find that $T' = 6.146 \times 10^7$ J/kg.
The initial velocity associated with this kinetic energy is given by $v' = \sqrt{2T'} \approx 11.09$ km/s which is, as argued above, slightly less than the escape velocity calculated in part A of our solution.

C. The distance of the "zero gravity" point P from the center of the Moon is given by $r' = R(0) - r \approx 3.84 \times 10^7$ m. The work required to lift 1 kilogram from the surface of the Moon to P is

$$T'' = \int_{R(M)}^{r} \left(\frac{GM}{x^2} - \frac{GM}{(R(0)-x)^2}\right) dx = \left(\frac{-GM}{r} + \frac{GM}{R(M)}\right) - \left(\frac{GM}{R(0)-r} - \frac{GM}{R(0)-R(M)}\right) \qquad (3)$$

Evaluating T'' as we evaluated T', we find that $T'' = 2.584 \times 10^7$ J/kg. This is also the kinetic energy per kilogram that should be dissipated if the spacecraft is to touch down on the Moon's surface with zero velocity. We note in conclusion that the minimum lift-off velocity that will send the kilogram from the Moon's surface back to P is $\sqrt{2T''} \approx 2.273$ km/s.

Once we reach the Moon, we ought to do some science. One reason for going to the Moon might be to examine the materials on its surface. So, our next problem which comes in two parts suggests a project that we might pursue.

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Problem 2. (Doing Science on the Moon).

A. A sample taken from the surface contained 4.0 grams of potassium of which 0.01% was in the form of the isotope ${}^{40}_{19}K$ which decays into argon with a half-life of 1.2 x 10^9 years. When the sample was heated in a vacuum, 2.4 cubic centimeters of argon at a temperature of 20° C and a pressure of 101.3 kPa were released. Calculate from these data a minimum age for the Moon. Assume that all of the argon in the sample was produced by the decay of potassium.
B. Alpha particles were used to bombard a small region of the lunar surface near the site from which the sample of part A above was taken. The purpose of the bombardment was to identify the chemical elements in the Moon's soil. In one experiment, the maximum kinetic energy of the recoiling alpha particles was 37% of their incident kinetic energy. Identify the element with whose nuclei the alpha particles collided.

Solution.

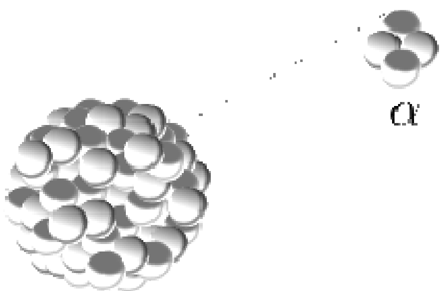
A. The 4 grams of potassium constitutes 0.1 mole which implies that there is 10^{-5} of a mole of ${}^{40}_{19}K$. Sample calculations based on the Ideal Gas Law, $Pv = nRT$, yield the number of moles of argon released:

$$(101.3 \times 10^3) \times (2.4 \times 10^{-6}) = n(8.31 \times 293) \rightarrow n = 10^{-4} mole$$

We can now use the equation of exponential decay

$$A(t) = A(0)e^{-\lambda t} \tag{4}$$

to compute the age of the Moon based on the assumptions that we have made. In this equation, $A(0)$ represents the amount of ${}^{40}_{19}K$ in the sample at $t = 0$ (the "instant" at which the Moon was formed). In our problem, $A(0) = 10^{-5} + 10^{-4} = 11 \times 10^{-5}$ mole. The parameter λ can be calculated from the half-life 1.2×10^9 years or radioactive potassium by noting that $\lambda = (\ln 2)/(1.2 \times 10^9) = 5.776 \times 10^{-10}$. Making the appropriate substitutions into equation (4) and letting $A(t) = 10^{-5}$, we solve the equation to find that $t = 4.15 \times 10^9$ years. This value is the estimated age of the Moon.



B. The maximum recoil energy will occur when an alpha particle suffers a head-on collision with the nucleus of an atom under bombardment. In this case, the approach and separation of the alpha particle and nucleus before and after collision will take place on a fixed line. Furthermore, we shall assume that, before collision, the nucleus was at rest.

We take the mass of the alpha particle to be 4 mass units and the mass of the nucleus that we wish to identify to be M mass units. We denote the velocity of the alpha particle before it collides with the stationary nucleus by v and the velocity of the nucleus after the collision by u. The Law of Conservation of Energy implies that

$$\left(\frac{1}{2}\right)4v^2 = (0.37)\left(\frac{1}{2}\right)4v^2 + \left(\frac{1}{2}\right)Mu^2.$$

Simplification yields

$$M = 2.52\left(\frac{v}{u}\right)^2. \tag{5}$$

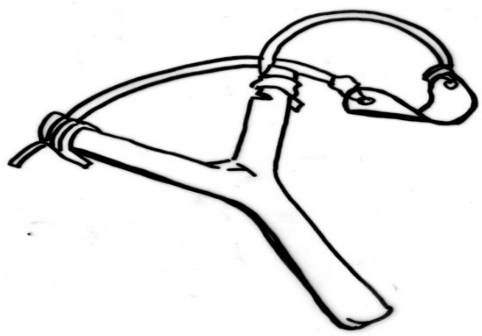
Since the kinetic energy of the alpha particle after the collision is 37% of the incident kinetic energy, its recoil velocity must be $\sqrt{0.37}\ v = 0.6087\ v$. Then the Law of Conservation of Momentum implies that

$$4v = Mu - 4(0.608)v\ .$$

This result in turn implies that

$$M = 6.432\left(\frac{v}{u}\right). \qquad (6)$$

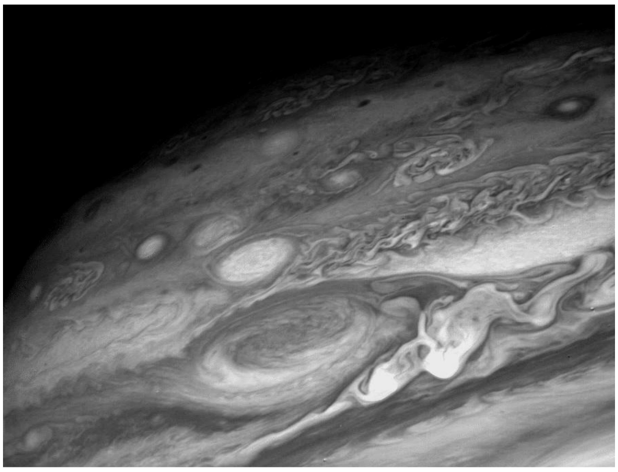
Solving equations (5) and (6) simultaneously yields $M = 16.4$ mass units. Thus we may infer that the element with which the alpha particle collided was oxygen.



Maximizing the efficient use of fuel in space is of prime importance. Robinson Crusoe's isolation would be as nothing compared to being lost in space. All ways of saving fuel ought to be investigated and exploited if possible. One concept is that of the "slingshot" principle. The idea is that when a spacecraft is traveling in the vicinity of a planet which is itself approaching the spacecraft (with respect to the rest frame of the fixed stars), a transfer of energy can occur which results in an increase in the speed of the spacecraft. Of course, the pull of the planet's gravitational field will slow the spacecraft down somewhat, but the velocity with which the spacecraft departs may still exceed the velocity with which it approached the planet.

A rough analogy may make the "slingshot" effect understandable to our armchair traveler. Let us imagine the level swing of a massive bat as it moves to strike an incoming ball. We take the ratio of the masses of the bat and ball to be comparable to the ratio of the masses of the spacecraft and planet. That is, the mass of the bat may be taken to be infinitely great. Let us assume that the collision between the bat and ball is perfectly elastic. Then, in the rest frame of the bat, the ball will rebound without loss of kinetic energy. The velocity v of the ball will be reversed in direction by its collision with the bat, but the magnitude of the ball's velocity will remain constant. Returning to the rest frame fixed on the baseball diamond, we see an increase in the magnitude of the velocity of the ball as its speed before the collision is reduced by the speed of the bat while its speed after the collision is increased by the same amount.

Thus the speed of the ball is increased by $2u$ where u is the speed of the bat. The forces developed during the collision were contact forces. In the interaction between the spacecraft and the planet, the forces are gravitational and contact does not occur since a crash landing would be undesirable. However, the basic idea is the same for the ball-bat and spacecraft-planet interactions.



Problem 3. (Getting a Lift).

A spaceship with a speed of v = 15 km/s approaches the planet Jupiter which is moving toward it at a speed of u = 13 km/s with respect to the fixed stars. The approach is oblique and at a distance so great that gravitational attractions may be neglected. The path of the spacecraft makes an angle of 30 degrees with the direction of the velocity of the oncoming planet. After being attracted and deflected by the planet, the spacecraft emerges from Jupiter's influence with its course altered by 300 degrees as it heads away at the angle of 30 degrees on the other side of Jupiter's direction. Find the gain in speed acquired by the spacecraft during its interaction with the planet.

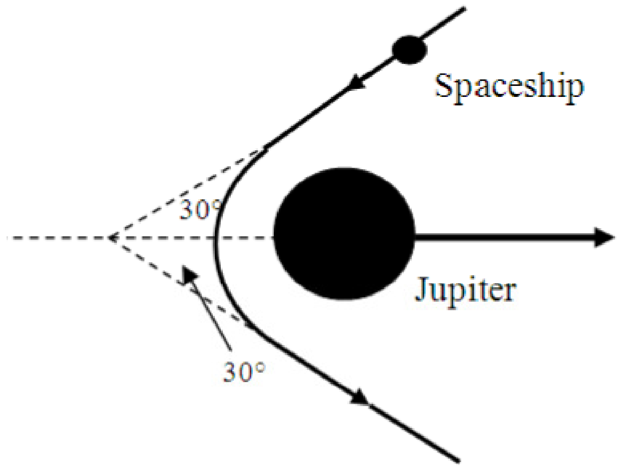


Figure 1. Jupiter's "Slingshot".

The geometry of the problem is suggested by Figure 1.

Solution.

Resolve the initial velocity of the spaceship into components parallel and perpendicular to the planet's path. Then

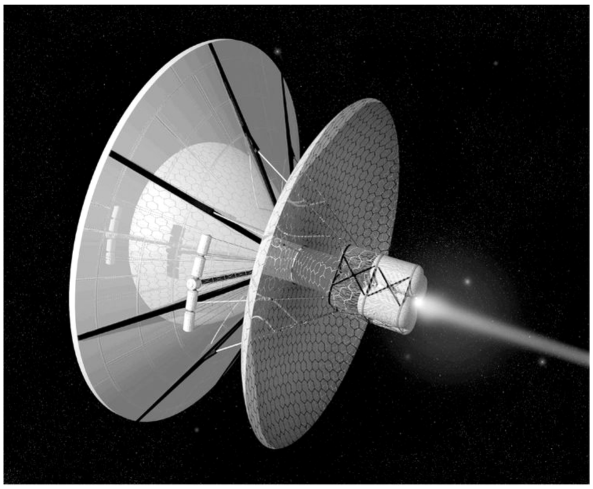
v(perpendicular, 0) = 15 sin 30° = 7.5 and v(parallel, 0) = 15 cos 30° = 12.99 .

The component of velocity perpendicular to the planet's path will be unaffected by the interaction between the spaceship and the planet. By the argument advanced in our analogy of the batted ball, the direction of the parallel component of velocity will be reversed and its magnitude increased by 2(13) = 26. The components of velocity as the spaceship heads away from Jupiter will be

v(perpendicular, 1) = 15 sin 30° = 7.5 and

v(parallel, 1) = -15 cos 30° – 26 = -38.99 .

The gain in speed will be $\sqrt{7.5^2+38.99^2}-\sqrt{7.5^2+12.99^2}=24.71\ km/s$. Thus Jupiter's "slingshot" increases the spaceship's speed by almost 25 km/s without the expenditure of any fuel at all!



The armchair imagination is completely free. So here is a really far out idea for saving fuel. Might not an interstellar spaceship collect protons in space as fuel? The protons would be fed into a fusion reactor to generate kinetic energy to accelerate the ship. Those interested in collecting protons should know that their density in space is n = 0.1 proton per cubic centimeter which is 10^5 protons per cubic meter.

Problem 4. (Accelerating the Spaceship).

How large a scoop would have to be designed to collect enough protons to develop an acceleration a = 10 m/s^2 for a 3500 tonne ship if f = 0.01 of the energy of the fusion reactor is converted to kinetic energy of the spaceship? Assume that 100% of the kinetic energy produced is converted to thrust. For protons to be captured they must incident on the surface of the scoop mechanism.
We need to recall that a mass of 1 tonne is M = 1000g, the mass of a proton is m = 1.7 x 10^{-27} kg, and the speed of light is c = 3 x 10^8 m/s in order to complete our calculations.

Solution.

Let us suppose that the velocity of the spaceship is v and that trailing behind the spaceship is the scoop which has a cross-sectional surface area A presented perpendicular to the direction of the spaceship's travel. The volume of space swept for protons per second by the scoop will be Av and the mass of protons collected per second will be $nmAv$ = 1.7 x 10^{-22} Av kg/s. Using Einstein's relationship for the energy equivalence of mass, we find that the energy collected is $1.7 \times 10^{-22}\ Avc^2 = 1.53 \times 10^{-5}\ Av$ J/s. This result represents an input of power. Of this input, $f(1.53 \times 10^{-5}\ Av) = 1.53 \times 10^{-7} Av$ J/s is converted to thrust.

The power required to develop the thrust to give the ship an acceleration of $a = 10$ m/s^2 is $3500Mav = 3.5 \times 10^7\ v$ J/s. Equating the two values of power just obtained, we find that

$$1.53 \times 10^{-7}\ Av = 3.5 \times 10^7\ v$$

implying that $A = 2.29 \times 10^{14}$ square meters, an enormous surface area indeed. If the scoop were circular, its radius would be $\sqrt{(2.29 \times 10^{14})/\pi} = 8.538 \times 10^6$ meters = 8537 km. Although the idea was an interesting one, such a scoop would be completely impractical.

Our armchair astronaut was a bit disappointed that his idea turned out to be unrealistic. However, he was not one to give up so easily. He had read about "wormholes" that might connect distant parts of the universe. Perhaps he could find an entrance. And then, in just the briefest of moments before it closed, he could ⋯