Oscillators Everywhere

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Abstract
We present a collection of problems about simple harmonic motion. These problems target students who are preparing for the Asian Physics Olympiads, various national Physics Olympiads, and the International Physics Olympiads. We hope that students will find the problems interesting and useful.

1. Introduction

Physicists have given us a model of the world filled with vibration at all scales in the universe from the smallest, grainiest quantum levels to the vast objects of study in astrophysics which demand a unification of general relativity and quantum mechanics.

The domain of classical physics is the "reasonable" world that we experience directly every day. The objects treated by classical physics are neither too small nor too large and their velocities are comfortably less than the speed of light. That domain might be called the "Goldilocks World" because everything therein is "just right." The ubiquity of vibratory motion holds throughout the "Goldilocks World" in which we live, and the first and often most sensible approximation of classical vibratory motion is simple harmonic oscillation, the projection of uniform circular motion onto a diameter. We suspect that more ink has been expended in writing "F = ma" and variants thereof than in writing any other equation of classical physics. A strong case could be argued that "$m(d^2x/dt^2) = -k\ x$" accounts for a considerable portion of that ink. The bobbing of a cork in water, the beating of the wings of birds and insects, the swinging of a pendulum, and even the tremors of the ground that we stand upon in the shock and after-shock of earthquakes are examples of vibrations that we have or may have observed in daily life. Since physicists seem to prefer "oscillation" as a word to "vibration", we feel that the title of our article is justified.

But let us stop for just a moment on our way to our first problem in order to comment on the vibratory motion induced by an earthquake. We urge readers to turn to the Internet to find information about the Yasaka Pagoda in Kyoto, Japan. The first version of the structure was built before 600 CE. From time to time during its history, the pagoda caught fire, burned and was rebuilt. The present structure dates from 1440. The design of the Pagoda which the citizens of Kyoto and visitors to that city see today serves as a model for "earthquake proof" construction. The five stories of the Pagoda are attached independently to a tall, central, vertical pillar. The response of the Pagoda to the energy that it absorbs with the shock of an earthquake is that each of the five levels moves freely about the central pillar with a damped oscillation as the energy is transmitted through the pillar to the ground and reabsorbed there.

The central supporting pillar was first incorporated into the design during a reconstruction a little before 940. Thus builders of long ago took advantage of their understanding of vibratory motion.

2. The Problems

Now, on to the problems! We shall present four problems. We shall give solutions to the first three but simply state the fourth, leaving its solution as a challenge and source of pleasure for our readers.

<u>Problem 1</u>. You are sitting in your garden sipping orange juice on a nice summer afternoon. Let us imagine a bee in search of just the right flower. Having spotted a nectar-laden blossom, the bee hovers above before settling down to gather the sweet, raw material for the hive's production of honey. As the bee hovers in midair, it is the motion of the bee's wings that produces the familiar buzz which we hear. Estimate the frequency of the bee's buzz.

<u>Solution</u>. Let the surface area of each of each of the bee's wings be denoted by A. Let the maximum amplitude of the flapping motion of the bee's wings be denoted by z_o. Let us denote the angular frequency of flapping by ω sec^{-1} and finally, let us denote the density of the air by ρ and the mass of the bee by m.

To help our thinking along, we draw a figure to suggest the bee hovering above a flower.

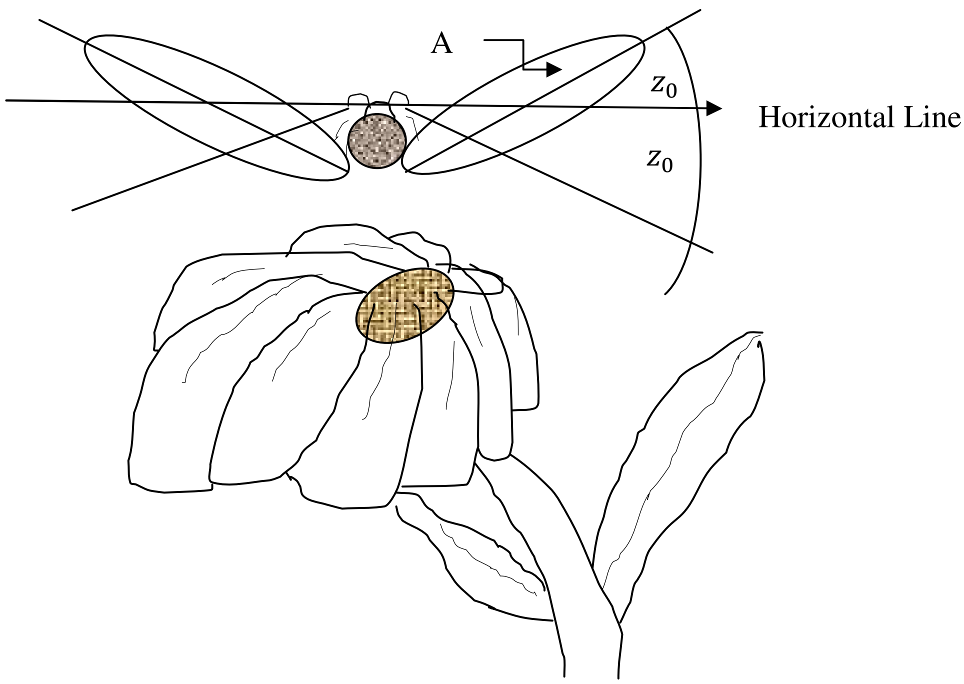

Figure 1. The Bee and the Flower

We must compute the upward force exerted by the air as it supports the weight *mg* of the bee. To do so, we compute the force exerted on the air by the bee during the downward stroke of its wings. The opposite reaction force by the air over the surfaces of the wings supports the weight of the bee.

The volume of air moved by each complete downward stroke of both wings is $2(2z_o)(kA)$ where k is a geometric constant depending on the shape of the region of space swept out by one complete downward stroke of each wing. The first factor of 2 is needed because there are two wings in motion. The second factor of 2 is needed because the tip of each wing travels twice the amplitude with each flap from top to bottom.

Sensible arguments may be advanced that $k \approx \frac{1}{2}$. In any event, we make that assumption and take the volume of air displaced by a complete down stroke of both wings to be $2z_oA$. Then the mass of the displaced air becomes

$$M_{air} = 2\rho A z_0.$$

The assumption that the motion of the wings is simple harmonic is equivalent to assuming that the displacement of each wing may be written as a function of time in the form

$$Z(t) = z_0 \sin \omega t.$$

The acceleration of each wing is then

$$\frac{d^2z}{dt^2} = -z_0\omega^2 \sin \omega t.$$

The magnitude of the force exerted by the bee on the air below it may be approximated by

$$\left| m_{air} \frac{d^2z}{dt^2} \right| = 2\rho A z_0^2 \omega^2 \sin \omega t \qquad (1)$$

with $0 \le \omega t \le \pi$ since we claim that the bee is not affected by the air during the half-period of the upward stroke if its wings.

By Newton's Third Law of Motion, the magnitude of the force that supports the bee during the down stroke will also be given by equation (1). Over the complete period of the motion of the wings, the supporting force is

$$F = \begin{cases} 2\rho A z_0^2 \omega^2 \sin \omega t & during\ the\ down\ stroke \\ 0 & during\ the\ up\ stroke \end{cases}$$

Next we compute the average value of the supporting force F over a complete period $\frac{2\pi}{\omega}$. We find that.

$$F_{avg} = \frac{\int_0^{\pi/\omega} 2\rho A z_0^2 \omega^2 \sin \omega t \, dt}{(2\pi/\omega)} = \frac{2\rho A z_0^2 \omega^2}{\pi}.$$

We must now make yet another assumption which is that z_o is comparable to the dimensions of the bee's wings. We accomplish that by assuming that $z_o = \sqrt{A}$ so that we may write that

$$F_{avg} = \frac{2\rho A^2 \omega^2}{\pi}.$$

Finally, we equate the average supporting force to the weight of the hovering bee. We obtain

$$\frac{2\rho A^2\omega^2}{\pi} = mg$$

or

$$\omega = \sqrt{\frac{\pi mg}{2\rho A^2}}.$$

We conclude by computing the approximate angular frequency ω for reasonable values of the parameters in our problem. Let m = 0.001 gm, A = 0.170 cm^2, ρ = 0.0013 gm/cm^3, and g = 980 cm/sec^2. We find that $\omega \approx$ 200 sec^{-1} which agrees well with observed buzz frequencies of 200 to 250 sec^{-1}.

Problem 2. Here is a familiar problem. A wooden disc bobs at the surface of an otherwise still pond. The dimensions of the pond are so great with respect to the dimensions of the disc that the level of the water surface remains unchanged as the motion of the disc proceeds. The buoyant force exerted by the water exceeds the weight of the disc when the disc is at its greatest depth and is less than the weight of the disc when it is at the top of its bobbing trajectory. When written, the equation of motion of the disc is easily seen to be that of simple harmonic motion.

Let us now look at a more complicated problem. That problem involves a solid wooden disc of radius a, thickness t, and uniform density ρ_{wood} which floats in a liquid which is contained in a vessel having dimensions comparable to the dimensions of the disc. When the disc is displaced from its equilibrium position, the resulting change in the level of the liquid surface may not be ignored.

Let us say that the density of the liquid is ρ_{liq} and that the vessel containing the liquid is a deep circular cylinder of radius R. We emphasize that we do not assume a << R.

Suppose that the disc is pushed down so that its horizontal upper surface is slightly above the level of the liquid surface and that the disc is then released from rest. Thereafter, the disc bobs up and down without damping by frictional effects while remaining upright at all times.

Determine the angular frequency of the vertical oscillations of the disc.

Solution. Let us say that the volume of the liquid in the cylindrical vessel is V_o and that the depth of the liquid would be h_o if the disc were not floating and bobbing in the liquid. Figure 2a

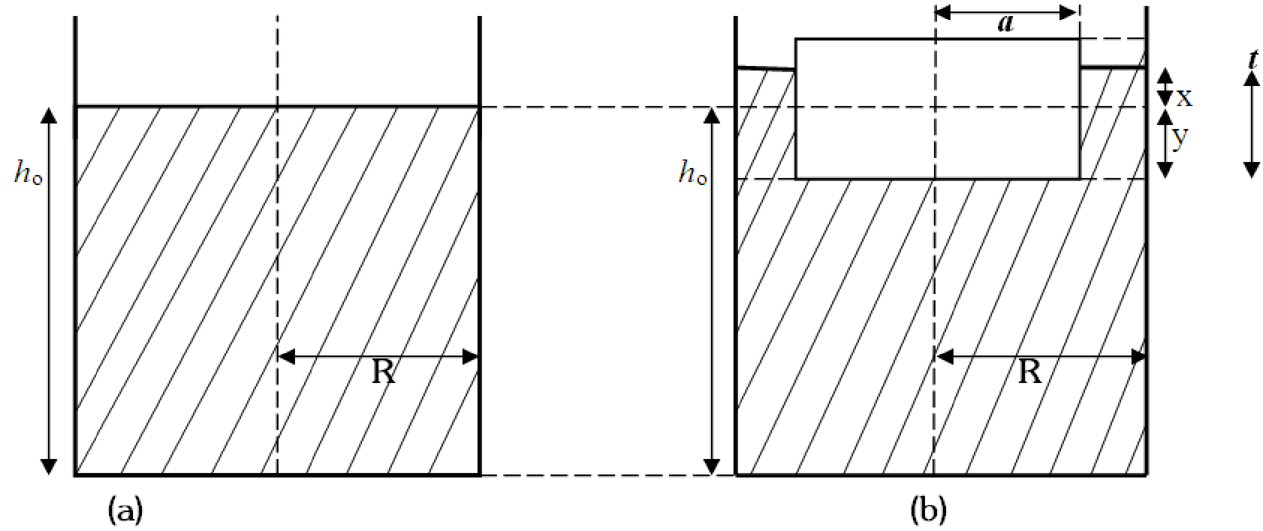

Figure 2. The Geometry of the Bobbing Disc.

shows the cylindrical vessel filled with the liquid to a depth of h_o before the disc is inserted. Figure 2b shows the vessel with the disc displacing a volume of liquid equal to the volume of the part of the disc which is now below the higher surface level of the liquid. We have drawn the Figure as though the central axes of the disc and the cylindrical vessel coincide. As the disc bobs upward, the liquid level descends; as the disc bobs downward, the liquid level rises. As shown in Figure 2b, the horizontal surface of the liquid is x units above h_o and the lower surface of the disc is y units below h_o.

We begin our computations by relating x and y. We note that the volume V of the liquid is not changed by inserting the disc. We may write that $V = R^2 h_o$ (see Figure 2a) and that
$V = \pi R^2(h_o + x) - \pi a^2(x + y)$ (see Figure 2b). Therefore,

$$\pi R^2 h_o = \pi R^2(h_o + x) - \pi a^2(x + y)$$

which implies that $0 = R^2 x - a^2(x + y)$ or

$$x = \frac{a^2 y}{R^2 - a^2}. \tag{2}$$

Next, we analyze the vertical forces acting on the disc and write an equation of motion. The varying upward buoyant force F_o exerted by the liquid on the disc is the weight of the liquid displaced by the disc. That force is $F_B = g\,\rho_{liq}(x + y)\,\pi a^2$ where g is the acceleration due to gravity. The constant downward force acting on the disc is its weight given by $W = g\,\rho_{wood}\,t\pi a^2$.

The magnitude of the unbalanced vertical force acting on the disc is

$$|F_B - W| = \left|\pi a^2 g\left[\rho_{liq}(x + y) - \rho_{wood}t\right]\right|.$$

The mass of the wood is $\rho_{wood}\ a^2 t$. Since y is the displacement of the lower surface of the disc from a fixed level, $\frac{d^2y}{dt^2}$ represents the acceleration of the disc in a fixed frame of reference.

Since the direction of the unbalanced force is obviously opposite to the displacement of the disc, we may now apply Newton's Second Law of Motion to write an equation of motion:

$$\rho_{wood}\pi a^2 t\frac{d^2y}{dt^2} = -\pi a^2 g\left[\rho_{liq}(x+y) - \rho_{wood}t\right].$$

Simplification yields

$$\frac{d^2y}{dt^2} = -\frac{g\rho_{liq}}{t\rho_{wood}}\left[x + y - \left(\frac{\rho_{wood}}{\rho_{liq}}\right)t\right].$$

Recalling that equation (2) gives *x* as a function of *y*, we rewrite the equation of motion as

$$\frac{d^2y}{dt^2} = -\frac{g\rho_{liq}}{t\rho_{wood}}\left[\left(\frac{a^2}{R^2-a^2}+1\right)y - \left(\frac{\rho_{wood}}{\rho_{liq}}\right)t\right].$$

Further simplification yields

$$\frac{d^2y}{dt^2} = -\left(\frac{g\rho_{liq}}{t\rho_{wood}}\right)\left(\frac{R^2}{R^2-a^2}\right)\left(y - \left(\frac{t\rho_{wood}}{\rho_{liq}}\right)\left(\frac{R^2-a^2}{R^2}\right)\right) . \qquad (3)$$

Finally, we make the change of variable

$$u = y - \left(\frac{t\rho_{wood}}{\rho_{liq}}\right)\left(\frac{R^2-a^2}{R^2}\right)$$

and transform equation (3) into an equation for simple harmonic motion. We now have

$$\frac{d^2u}{dt^2} = -\left(\frac{g\rho_{liq}}{t\rho_{wood}}\right)\left(\frac{R^2}{R^2-a^2}\right)u.$$

The motion is simple harmonic in the variable *u* with *y* oscillating about its equilibrium value of $\left(\frac{t\rho_{wood}}{\rho_{liq}}\right)\left(\frac{R^2-a^2}{R^2}\right)$. The desired angular frequency of the oscillation is

$$\omega = \sqrt{\left(\frac{g\rho_{liq}}{t\rho_{wood}}\right)\left(\frac{R^2}{R^2-a^2}\right)}.$$

To get a sense of the numbers involved in our problem, we let the disc bob in water in a cylindrical vessel or radius R = 8 cm. We suppose that the disc is made of white oak and has radius a = 5 cm and thickness t = 4 cm. Water has density 1 gm/cm^3 and the white oak used for the disc has the high density of 0.77 gm/cm^3. We assume that g = 980 cm/sec^2. Evaluating ω for the values given, we find that the angular frequency would be 22.85 sec^{-1} which implies a vibrational frequency of $\frac{\omega}{2\pi}$ = 3.64 sec^{-1}.

<u>Problem 3</u>. A right circular cone with altitude *h* and volume V has a base of radius *r*. The vertex angle is θ. The entire mass M of the cone is concentrated at its vertex. The cone is turned so that its vertex points downward and it is then placed in a liquid of density ρ > M/V so that it floats with its altitude aligned with the vertical direction.

The reader may imagine a light conical drinking cup floating in the liquid with a small but heavy metal weight resting at its vertex as suggested by Figure 3.1. We assume that, if the cone bobs up and down, the level of the liquid surface will not be affected and that viscous drag on the cone may be ignored. We also assume that there will be no horizontal perturbations of the cone as it oscillates along the vertical direction.

We ask first for a differential equation of motion for the vertex of the cone as it bobs up and down. Since this equation will turn out to be nonlinear, we also ask for a linear approximation to this equation and for the corresponding simple harmonic frequency for oscillations of very small amplitude.

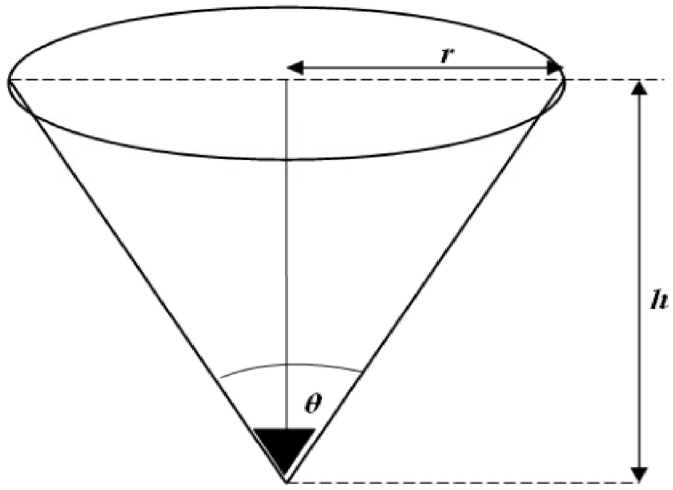

Figure 3.1 The Right Circular Cone.

Solution. We begin by computing the equilibrium depth of the floating cone. We denote the depth of the vertex below the liquid surface by y and the radius of the circular cross section of the cone at the liquid surface by x. We show the geometry of the bobbing cone in Figure 3.2.

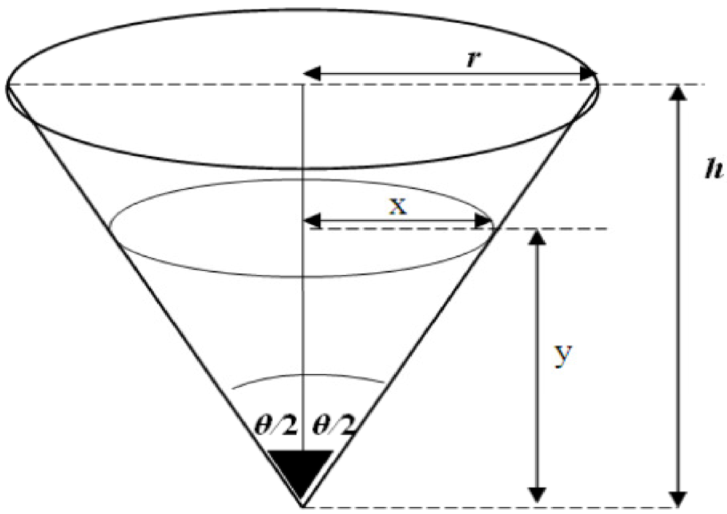

Figure 3.2 The Geometry of the Bobbing Cone.

Let y_0 denote the depth of the vertex when the cone is floating at rest in equilibrium. The radius x will have the corresponding value x_0 where $\frac{x_0}{y_0} = \tan\frac{\theta}{2}$. Thus $x_0 = y_0 \tan\frac{\theta}{2}$.

At the equilibrium position, the weight of the liquid displaced by the cone must equal the weight of the cone Mg where g is the acceleration due to gravity. Thus

$$Mg = \frac{\pi x_0^2 y_0}{3}\rho g = \frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)y_0^3 g}{3}$$

which implies that

$$y_0 = \left[\frac{3M}{\pi\rho\, tan^2\frac{\theta}{2}}\right]^{1/3}. \qquad (4)$$

The last calculations may be repeated in deriving the equations of motion. As the cone bobs up and down, the upward buoyant force is given by

$$F_B = \left(\frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)g}{3}\right)y^3$$

where y varies as opposed to remaining fixed at y_0 while the weight of the cone keeps the constant value Mg. Taking the downward direction to be positive for y, the equation of motion becomes

$$M\frac{d^2y}{dt^2} = Mg - \frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)gy^3}{3} \qquad (5)$$

Let us recall that in the calculations leading up to equation (4), we showed that $Mg = \frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)gy_0^3}{3}$. This recollection allows us to rewrite equation (5) as

$$M\frac{d^2y}{dt^2} = \frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)g}{3}(y_0^3 - y^3) = \frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)g}{3}(y_0 - y)(y_0^2 + y_o y + y^2).$$

In seeking a simple harmonic approximation of the bobbing cone, we take $|y_0 - y|$ to be very small so that $y \approx y_0$. Thus our last equation becomes

$$\frac{d^2y}{dt^2} = \frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)g}{3M}(y_0 - y)(3y_0^2) = \frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)gy_0^2}{M}(y_0 - y).$$

Letting $u = y - y_0$, we may write that

$$\frac{d^2u}{dt^2} = -\frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)gy_0^2}{M}u.$$

The angular frequency for simple harmonic approximation is

$$\omega = \frac{1}{2\pi}\sqrt{\frac{\pi\rho\left(tan^2\frac{\theta}{2}\right)gy_0^2}{M}}.$$

Substituting the expression for y_o given by equation (4) into the equation for ω and simplifying the result yields

$$\omega = \sqrt{g\left(\frac{9\pi\rho tan^2\frac{\theta}{2}}{M}\right)^{\frac{1}{3}}} = \left(\frac{9\pi\rho g^3 tan^2\frac{\theta}{2}}{M}\right)^{\frac{1}{6}}.$$

Then the frequency of vibration for the cone in its simple harmonic approximation is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\left(\frac{9\pi\rho g^3 tan^2\frac{\theta}{2}}{M}\right)^{\frac{1}{6}}.$$

Finally, let us imagine that the cone is bobbing up and down in water. To obtain a feel for the motion for which we have given a mathematical description, let us assign numbers to our symbols and determine the frequency *f*.

The density of water is ρ = 1 gm/cm^3. Let θ = 30°, M = 200 grams, and g = 980 cm/sec^2. We find that f ≈ 2.32 sec^{-1}.

Problem 4. For our Readers

(1992 British Physics Olympiad)

A light pulley of radius R has half of its curved surface covered by a uniform strip of metal sheet of total mass M as indicated in Figure 4. A cord is wound round the pulley and one of its ends is attached to a mass *m* that hangs vertically.

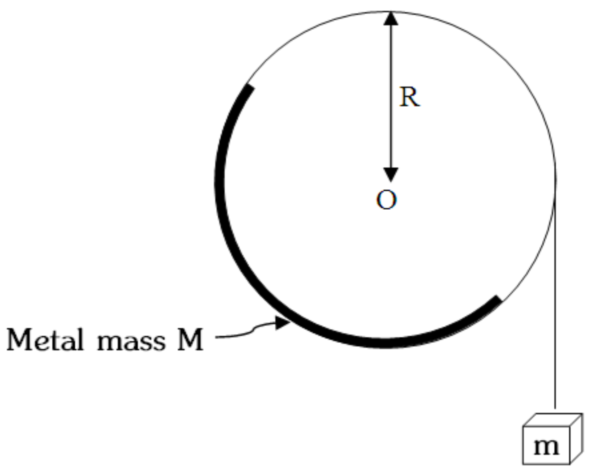

Figure 4. The Pulley with the Strip of Metal and Mass *m* attached to the Cord.

Verify that the distance of the centre of mass X of the metal strip from the axis O of the pulley is $(2R/\pi)$.

Determine each of the following:

The equilibrium inclination, θ, of OX to the vertical for all values of m for which equilibrium is possible.

The values of m for which equilibrium is not possible.

The period of oscillation of the system, for small angular displacements from the stable equilibrium orientation θ, using energy considerations, or the equation of motion.

By means of displacement-time graphs for m, describe its possible modes of motion.

Note that for small $\alpha, \cos(\theta + \alpha) = \cos\theta - \alpha\cos\theta - \frac{\alpha^2}{2}\cos\theta + \cdots$