A few good orbits

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Abstract
We present three novel problems based on Kepler's laws, which help in bringing out various facets of orbital dynamics. The problems were designed by the authors for the training and selection camp of the Indian National Astronomy Olympiad 2008. The problems require only pre-calculus mathematics and high-school physics, but demand strong visualization of the scenarios presented. By constructing fun problems involving elaborate situations like the ones presented here, students are exposed to practical applications of the Kepler's Laws like satellite orbits.

1. Introduction

The Indian National Astronomy Olympiad program started in the year 1999. The Olympiad competitions typically involve pre-university students who compete through various stages finally leading to selection of national team for respective international Olympiads. Typical enrollment for the first stage of Astronomy Olympiad in India is close to 15,000 students. As pre-university curriculum in India does not include advanced topics in astronomy and astrophysics, initial selection tests focus on the students' understanding of physics, mathematics (pre-calculus) and general mental ability. The last stage of selection is a 3 week summer camp for nationally selected top 50 students who are exposed to various topics in astronomy and astrophysics, involving nontrivial physical arguments and pre-calculus mathematics. Designing problems for the selection tests during the camp is a challenging task as the problems are expected to be unconventional and yet within boundaries of student's scholastic competencies. Over the years a number of fairly sophisticated problems in various fields of astronomy have been posed in the various levels of the Olympiad program leading up to the selection of the national team. To level the playing field for all participants who may have prepared in different ways prior to the selection camp, all problems in selection tests are newly designed (i.e. none of them are pre-published) by the Olympiad resource persons, who include teaching faculty as well as past Olympiad medalists. What follows is an analysis of three problems designed by the authors in the field of orbital dynamics which were posed in the 10th Indian National Astronomy Olympiad camp in the year 2008.

2. A Brief Overview of Orbital Mechanics

The basic orbital mechanics is governed by just two equations: Newton's law of gravitation and Newton's second law of motion. The first provides an approximate expression for the force due to gravitation between two bodies. The second relates this force to the acceleration of the body enabling us to solve the kinematics of the bodies involved. Based on these two equations, a complete solution for the two body problem is well-known as a one-body problem with a reduced mass plus the free motion of the center of the mass. We can approximate the reduced mass to the mass of the bigger body (called the central body), if it is many times as massive as the second (orbiting) body. For this special case, lets us summarize necessary terminologies, laws etc.

1. Kepler's Laws:
    a. All orbits are conic sections with the central body at one of the foci (let us call this as the prime focus).
    b. The area of the sector, centered at the prime focus, traced by the orbiting body is proportional to the time taken to cover it.
    c. The square of the time period of the orbiting body is proportional to the cube of the semi-major axis of its elliptical orbit.

2. Peri- and Ap- position: At some point in its orbit the orbiting body must be at its closest distance to the central body. This is peri-position. Likewise, the furthest distance is the ap-position. Obviously for unbound obits the ap-position does not exist (i.e. it is at infinity).
3. True-anomaly(V): The angle subtended by the radius vectors from the prime focus to the peri-position and the current position of the orbiting body.

The understanding of Kepler's Laws and teaching the topic in an elegant manner has been an important topic of discussion amongst physics education community [1], [2], [3], [4], [5], [6], [7], [9][10].

3. The Solution and the Analysis of the problems

What follows are the actual statements of the problems and their solutions.

3.1. Problem 1 : Celestial Spheres for the Extra-Terrestrial

3.1.1. Problem Statement:

4N1K37-2 is launched from the Earth into a circular geosynchronous orbit and further it is ensured that the phase of the Earth in the space shuttle's sky remains constant The space shuttle's transmission antenna always points towards the Earth and the solar panels always face the sun. The solar panels were built with an inherent gravitational field such that people can stand on them. The space shuttle began its transmission to the ground station (somewhere on the Equator) when it was in the Equatorial Plane of the Earth on summer solstice day.

At the beginning of the transmission, an observer Alice who is at the ground station could see the space shuttle on her zenith and the sun on her western horizon. Suppose an observer, Bob, standing on the solar panel of the space shuttle is facing the Earth. Find the constellation in the anti-Earth direction of the observer (on the space shuttle) after exactly 100 (sidereal) days of transmission.

3.1.2. Motivation:

It is often possible to confuse between a geostationary and a geosynchronous orbit. In the case of a geo-synchronous satellite all that is required is for the period to be equal to one day. In other words, for a fixed observer on the ground, it should return to the same position in the sky at the same time every day. The geo-stationary orbit is special case of the geo-synchronous orbits, where the orbit is equatorial and circular. More importantly, the problem demands strong spatio-visual thinking that is generally neglected in science teaching, especially in India.

3.1.3. Solution:

The problem inherently is very simple. All that is needed is a clear mental picture. This is the step by step procedure to identify the orbit:

i. Firstly we note that the orbit is a geosynchronous orbit. That means, after 24 hrs the satellite returns to the same position relative to the earth. Thus the length of the semi-major axis (a) of the orbit is fixed.
ii. The orbit is given to be circular. Hence the eccentricity (e) is also fixed to be 0.
iii. Next we find the inclination (i). It is given that in the satellite sky, Earth has constant phase. If the Sun-Earth-Satellite angle is θ at a given moment that after 12 hours it will be $(180° - \theta)$. Thus the phase angle would change unless of course θ were maintained to be 90°. The orbit which does this is the one perpendicular to the ecliptic and parallel to the terminator.
iv. Now, we must find the initial position in the orbit. It is given that the first transmission occurs on summer solstice day. Thus the Earth-Sun vector points to an ecliptic longitude of 90°. Now the satellite can have 2 positions in this orbit. One such that the Earth-Satellite vector point to a longitude of 0° and the other such that it points towards 180°. The fact that observer Alice saw the satellite at her zenith and also saw the Sun in the western sky at the same time implies that the Earth satellite angle is in fact 180° ecliptic longitude.
v. As for the direction of revolution we could have either but the fact that the second position is asked after exactly 100 sidereal days renders the point irrelevant.

Now we note that initially the Anti-Earth direction was the autumnal equinox point. After 100 days which correspond to approximately 3 and $\frac{1}{3}$ months, the constellation will shift by three zodiac constellations. Initially the anti-Earth side was Virgo and finally it turns out to be Sagittarius. (Note: Rough Calculations show that the ecliptic longitude turns out to be 278)

3.1.4. Remarks:

Although the problem appeared to be highly calculation oriented, no actual calculation was required to arrive at the first order solution. All that is needed is a step by step analysis aided by strong visualization and the realization that the geosynchronous nature of the orbit is the key input.

3.2. Problem 2 : A Tale of two Phases

3.2.1. Problem Statement:

In the year 2021 the ISRO's home-bred space shuttle identified by the code MJ-3105 is planned to dock with our space observatory 4N1K37 which orbits the earth at 600 km in circular orbit. Imagine, that due to some slight calculation errors the MJ lands in the correct orbit but at a wrong phase such that the distance between MJ and the observatory is 916 km and the observatory is ahead of MJ in its orbit. (Assume the orbit of 4N1K37 to be a non-precessing circle). As a corrective measure an appropriate elliptical orbit is suggested. If MJ-3105 is constrained by the fact that it can only apply thrust in a direction tangential to its orbit,

i. What possible orbits can it be put into such that after one complete periodof MJ in its new orbit, its location and the observatory's location will turn out to be the same? Draw a neat diagram depicting the same. (Note: After one complete revolution the corresponding thrust in the opposite direction will restore the circular orbit, thus completing this phase correction sequence.)

Figure for problem 2

ii. Find the minimum thrust required for this complete maneuver and also find the corresponding orbit.

iii. If the initial phase angle difference was 30° instead, what would be the minimum thrust?

3.2.2. Motivation:

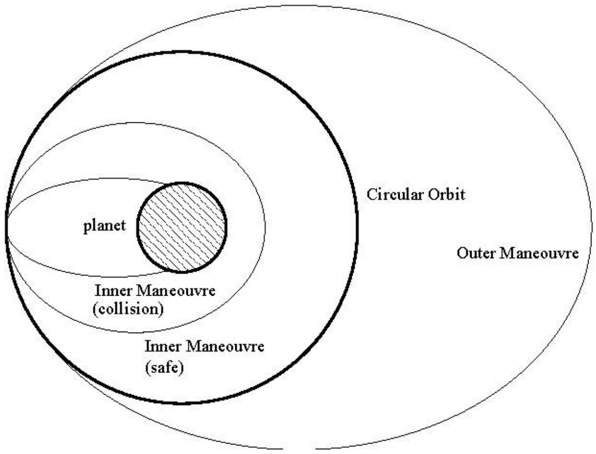
Inorder to change the phase of an object in orbit around a central body one needs to exert some sort of external force on it. This force will change the momentum of the body at that given point in the orbit. Now as a result of this new velocity the orbit itself changes defeating the whole purpose of the initial momentum. The solution to this Catch-22 situation is the selection of certain specific orbits set out in this problem.

3.2.3. Solution:

Consider the figure below. Now we know that the initial straight line distance between 4N1K37 and MJ-3105, with both in same circular orbit is 916km.

We can calculate the chord length as,

$$d = 2\sin(\theta/2) \text{ i.e. } \theta = 2sin^{-1}\left(\frac{d}{2r}\right)$$



Now let TMJ be the time taken by MJ to complete one orbit. We know by Kepler's third law that inner orbits have smaller time period and that the outer orbits will have a larger time period. Consider the orbits shown in the figure1. For 4N1K37 and MJ to come to same position, when MJ is in an outer orbit, 4N1K37 must cover an integral number of orbits less an angle θ in one orbital period of MJ. However when MJ is in an inner orbit, it must cover an integral number of orbits in time required for 4N1K37 to cover an angle 2 θ in its orbit. Hence the governing conditions for circular orbit of 4N1K37 are

Figure Different maneuvering orbits discussed in problem 2.

$$T_{MJ,outer} = \left(\frac{2(n+1)\pi - \theta}{2\pi}\right) T_{4N} n\left(T_{MJ,inner}\right) = \left(\frac{2\pi - \theta}{2\pi}\right) T_{4N}$$

By Kepler's Third law,

$$\frac{a_{MJ}^3}{T_{MJ}^2} = \frac{R_{4N}^3}{T_{4N}^2} a_{MJ} = R_{4N}\left(\frac{T_{MJ}}{T_{4N}}\right)^{\frac{2}{3}}$$

Thus we get two cases for inner and outer orbits,

$$a_{MJ,outer} = R_{4N}\left(n + 1 - \frac{\theta}{2\pi}\right)^{\frac{2}{3}} = R_{4N}\left(n + 1 - \frac{1}{\pi}sin^{-1}\frac{d}{2r}\right)^{\frac{2}{3}}$$

$$a_{MJ,inner} = R_{4N}\left(\frac{1}{n}\left(1 - \frac{\theta}{2\pi}\right)\right)^{\frac{2}{3}} R_{4N}\left(\frac{1}{n}\left(1 - \frac{1}{\pi}sin^{-1}\frac{d}{2r}\right)\right)^{\frac{2}{3}}.$$

For an elliptical orbit, using energy conservation, velocity at a distance r from the focus is given by,

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}.$$

Hence thrust per unit mass is given by,

$$P=\sqrt{GM}\left(\sqrt{\frac{1}{r}}-\sqrt{\frac{2}{r}-\frac{1}{a}}\right)P_{outer}=\sqrt{\frac{GM}{R_{4N}}}\left(\sqrt{2-\left(n+1-\frac{1}{\pi}sin^{-1}\left(\frac{d}{2r}\right)\right)^{\frac{-2}{3}}}-1\right)P_{inner}$$

$$=\sqrt{\frac{GM}{R_{4N}}}\left(1-\sqrt{2-\frac{1}{n}\left(1-\frac{1}{\pi}sin^{-1}\frac{d}{2r}\right)^{\frac{-2}{3}}}\right).$$

Now note, for the outer orbits as the order of the orbit (n) increases, the thrust must increase as for the same perigee point the semi-major axis is increases. Similarly for the inner orbits as n increases for the same apogee point the semi-major axis is decreasing. Hence both the velocity difference and the thrust are more. Hence the order of both orbits should be 1. Explicitly calculating ratio of thrust to initial momentum values we see,

$$P_{outer,916km}=0.1681\ ,\ P_{inner,916km}=0.0071$$

$$P_{outer,30^\circ}=0.1627\ ,\ P_{inner,30^\circ}=0.0303$$

Thus it seems as though, both the inner orbits are favoured. As a final consistency check we calculate the ratio of the perigee distance to the radius of the earth. Note for inner orbits the current point is at apogee.

$$r_{perigee}=2a-r_{apogee}\rho=\left(2\left(1-\frac{1}{\pi}sin^{-1}\frac{d}{2r}\right)^{\frac{2}{3}}-1\right)\frac{R_{4N}}{R_e}\rho_{916km}=1.062\rho_{30^\circ}=0.970$$

Thus, in the second case, perigee turns out to be inside the Earth. Hence, even though the inner orbit is favoured due to low thrust, only the outer orbit can be chosen. Initial velocity is the circular velocity i.e.

$$v=\sqrt{\frac{GM}{R_{4N}}}\cdot 7.561\,km/s$$

Hence, we have $P_{inner,916km}=53.6\,m/s$ and $P_{outer,30^\circ}=1230\,m/s$

3.2.4. Remarks:

The key ideas to learn from the problem are:

i. If a body applies tangential force the initial point is constrained to be either the peri-point or ap-point.

ii. In order to change the phases of the orbit we can select a few quantized orbits such that the satellite can make up for the lost phase by going around once in a new orbit.

iii. For any orbital maneuver it is imperative to check whether the modified orbit collides with the central body. This check must be externally imposed and cannot be guessed from the initial conditions.

3.3. Problem 3: The Great Escape

3.3.1. Problem Statement:

An earth-like planet 'Echor', resides in a stellar system at approximately 10 A. U. from its parent star, which is similar to the Sun. The planet has no natural satellite and other planets in that system are far from this planet. The Echorans have decided to launch their first interstellar mission to colonize other habitable planets in other star systems. The starship 'Chabya' designed for the purpose has the following dimensions. It consists of two spherical sub-ships (H and B), approximately 20km in diameter each, attached to one another by a small tether in the center. The density of the spaceships is the same as that of the planet i.e. approximately 5.5 g/cc. The first step to get the massive Chabya to the closest star is to get it out of the gravitational influence of Echor. For this purpose the ship is firstly put up in a circular orbit about Echor with radius equal to12,960,500km. The ship is oriented such that the centers of the two sub ships and the center of Echor are always collinear and Chabya-B always lies on the inside. It is believed that when an appropriate retarding impulse is provided, the spacecraft will go into a new orbit and at somepoint, Chabya-H will be able to escape outward leaving Chabya-B behind. Find the minimum retarding impulse to achieve this. (Note: The breaking strength of the tether is $7\text{x}10^{12}$ N.)

3.3.2. Motivation:

The main idea behind this problem is that for a system of rigid bodies with no relative rotation, the velocity of each body is same as that of the center of mass. However, when the individual parts separate, they continue to have the same initial velocity but may move differently based on the differential forces acting on them. In addition, for a given velocity, the orbit depends on the gravitational potential at that point. Thus, for a small change in initial position, the final positions of the bodies may be remarkably different. Another key idea involved is that parameters of the problem are such that mutual gravitational force between the bodies is not to be ignored.

3.3.3. Solution:

Firstly, consider the two satellite system. The forces acting on each component are a net inward force due to both tension and gravity. In the frame of the center of mass of the system, there is a net outward force on each body. And a differential force due to gravity. Consider the force equation for Chabya-H,

$$F_{centrifugal} = F_{gravitational} + T_{net}$$

For Chabya-B,

$$F_{centrifugal} = F_{gravitational} - T_{net}$$

Now the tension is a combination of tether tension and gravitational interaction.

$$T_{net} = T_{gravitational} + T_{tether} = \frac{Gm^2}{(2r)^2} + T_{tether} = \frac{4G\pi^2 r^4 \rho^2}{9} + T_{tether}$$

Where, r is the radius of each sub-part.
In this orbit, we have $T_{net} = 1.41 \cdot 10^{15} N$.

Now note that the mutual gravitational force between the Chabya-B and the Chabya-H in the initial orbit is much larger than the limiting breaking strength of the tether. Almost the entire contribution to the Tnet comes from the mutual gravitational force between the two components. Hence the tether cannot automatically break in this orbit. To separate the two parts automatically, one must move the satellite to a point where their mutual gravitational attraction can be overcome; or in other words move it near the Roche Limit.

Now the Roche Limit is given by,

$$d_{Roche} = R_{planet} \left(\frac{2\rho_M}{\rho_m}\right)^{\frac{1}{3}} = R_{planet} 2^{\frac{1}{3}}.$$

We give a retarding impulse to the satellite such that it goes in an elliptic orbit with the initial orbit position as the ap-point and the Roche limit distance as the peri-point. We need not go any closer, as when the tether will be at the Roche limit, Chabya-B will be inside the Roche sphere and Chabya-H will be just outside and the tension in the tether will just exceed the breaking strength. On separation of the tether, both the parts will continue to have the center of mass velocity. As such Chabya-H which is outside may overcome the gravitational potential and escape whereas Chabya-B which is inside will fall inwards.

Once again, we use the expression for velocity of any orbiting body in an elliptic orbit

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}.$$

Hence to make the Roche limit distance as the peri-point, the velocity at the ap-point should be,

$$v_{apo} = \sqrt{GM\left(\frac{2}{a(1+e)} - \frac{1}{a}\right)} = \sqrt{\frac{2GM}{R_{apo} + R_{peri}}\left(\frac{R_{peri}}{R_{apo}}\right)} v_{apo} = \sqrt{\frac{2GM}{R_{circular} + d_{Roche}}\left(\frac{d_{Roche}}{R_{circular}}\right)}$$

For the initial circular orbit,

$$v_{circular} = \sqrt{\frac{GM}{R_{circular}}}$$

Hence we have,

$$\Delta v = v_{circular} - v_{apo} = \sqrt{\frac{GM}{R_{circular}}} - \sqrt{\frac{2GM}{R_{circular} + d_{Roche}}\left(\frac{d_{Roche}}{R_{circular}}\right)}$$
$$= \sqrt{\frac{GM}{R_{circular}}}\left(1 - \sqrt{\frac{2^{\frac{4}{3}}R_{planet}}{R_{circular} + 2^{\frac{1}{3}}R_{planet}}}\right).$$

Thus the change in momentum is given by,

$$\Delta p = m_{satellite}\sqrt{\frac{GM}{R_{circular}}}\left(1 - \sqrt{\frac{2^{\frac{4}{3}}}{\frac{R_{circular}}{R_{planet}} + 2^{\frac{1}{3}}}}\right).$$

Noting that it is an 'Earth-like' planet, we use values of the Earth mass and the Earth radius. The numerical value of this change is $3.2\ 10^{19}$ kg m/s.

3.3.4. Remarks:

(i) Although the gravitational force of everyday bodies is negligible, one must consider it when body sizes are substantially large and they are placed in a weak gravitational potential.

(ii) The reason why Chabya-H leaves is not clear, immediately, however careful analysis shows us that if the split between Chabya B and H occurs at the right spot then the two will separate and the initial momentum will carry one part out but the other will fall in.

4. Summary

The above problems are meant to give students a flavor of the vast and exciting field of Orbital Dynamics. The attempt was to show that in orbital mechanics, problems which look daunting at the first-sight can be solved with nothing more than essential simple physical arguments and knowledge of pre-college mathematics. It must be noted that each of these problems stimulates the student to think in an 'out-of-the-box' manner, and emphasizes concepts rather than mechanical calculations. Exposure to such problems will hopefully attract meritorious students to the wonderful world of Astronomy.

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